Effect of spatial reflection symmetry on the distribution of the parametric conductance derivative in ballistic chaotic cavities

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(Received 25 October 2004; published 4 March 2005)

We study the effect of left-right symmetry on the distribution of the parametric derivative of the dimensionless conductance T with respect to an external parameter X, $\partial T/\partial X$, of ballistic chaotic cavities with two leads, each supporting N propagating modes. We show that T and $\partial T/\partial X$ are linearly uncorrelated for any N. For $N=1$ we calculate the distribution of $\partial T/\partial X$ in the presence and absence of time-reversal invariance. In both cases, it has a logarithmic singularity at zero derivative and algebraic tails with an exponent different from the one of the asymmetric case. We also obtain explicit analytical results for the mean and variance of the distribution of $\partial T/\partial X$ for arbitrary N. Numerical simulations are performed for $N=5$ and 10 to show that the distribution $P(\partial T/\partial X)$ tends towards a Gaussian one when N increases.

I. INTRODUCTION

In recent years there has been great interest in coherent quantum transport in mesoscopic devices in both experiment and theory (for a review, see, Ref. [1]). In quantum systems whose classical dynamics is chaotic, phase coherence and quantum interference give rise to sample-to-sample fluctuations in most transport properties with respect to small perturbations in incident energy, cavity shape, or an external parameter such as an applied magnetic field. Therefore a statistical analysis by random-matrix theory (RMT) becomes applicable; the details of the problem being irrelevant, emphasis must be put on universal aspects such as the symmetries of the system. Then, time-reversal invariance (TRI) and spin-rotational symmetries [2], or the spatial symmetries present in the system, characterize their universal statistics [3–6].

Some experiments concerned the parametric dependence of the conductance study ballistic quantum dots connected by leads with few propagating modes to electron reservoirs, as happens in semiconductor systems [7–11]. The conductance is measured as a function of the magnetic field and the shape of the quantum dot (controlled by gate voltages). The derivative of the conductance with respect to an external parameter, which is analogous to the level velocity [12–15] of the dot, is a very important quantity that gives us the response of a mesoscopic sample to a small perturbation. Only the asymmetrical situation has been considered in the literature using an RMT approach [11,16–19].

Other methods to verify RMT predictions are given by wave scattering experiments, such as microwaves [20–22] and acoustic resonators [23]. In the microwave technique, for example, the conductance is measured as a function of the frequency of incidence, applied magnetic field, and cavity shape.

In these wave scattering experiments the external parameters are easier to control. The symmetric case can be constructed to analyze the strong effects of spatial symmetries on the conductance, predicted by RMT [3–5]. Experiments have been conducted on left-right symmetric cavities [22].

However, no theoretical results exist for the distribution of the conductance velocities. The purpose of this paper is to study the effect of the spatial left-right symmetry in the distribution of the parametric derivative of the conductance. Thus we consider a chaotic left-right symmetric cavity connected to two symmetrically located leads for two different symmetries, with and without TRI, in the absence of an applied magnetic field [24]. Other spatial symmetries [4] are not considered.

When TRI is broken by a magnetic field the problem of left-right symmetric cavities is mapped to the one of asymmetric cavities in the presence of TRI. That problem was studied by Brouwer *et al.* [19] using an RMT method to calculate analytically the joint distribution of the dimensionless conductance T and its velocities $\partial T/\partial V, \partial T/\partial X$, with respect to the gate voltage V and a external parameter X (typically a magnetic field), for a quantum dot with two single-mode point contacts. The statistical distributions $P(\partial T/\partial V)$, $P(\partial T/\partial X)$ show algebraic tails, and in the absence (presence) of TRI they have a cusp (divergence) at zero velocity, and the second moment is finite (infinite). RMT predictions have been verified by experimental results [11].

In the present work we perform analytical calculations for the one channel case to reduce to quadratures the distributions of $\partial T/\partial E$, and the derivative $\partial T/\partial X$ with respect to an external parameter X (shape deformation parameter in microwave cavities), using a scattering matrix formalism. By direct numerical integration we obtain $P(\partial T/\partial E), P(\partial T/\partial X)$. Analytically, we obtain their mean and variance and show also that $T, \partial T/\partial E, \partial T/\partial X$ are linearly uncorrelated for any number of channels. Besides, we obtain $P(\partial T/\partial E), P(\partial T/\partial X)$ by numerical simulations using a scattering matrix.

The formal elements such as the scattering matrix for the symmetric case, its energy, and parametric derivatives are introduced in the next section, as well as the joint distribution of them, for an arbitrary number of propagating modes N in the leads. The $N=1$ case is the subject of Sec. III A for both in the presence and absence of TRI. Section III B is dedicated to an arbitrary N case. Finally, the conclusions are presented in Sec. IV.
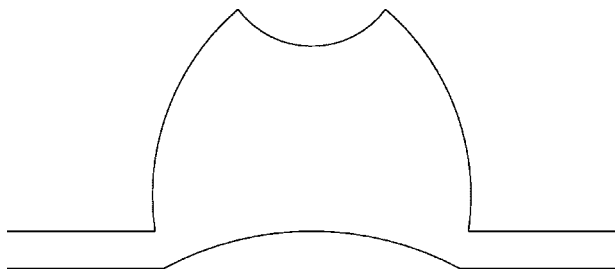
FIG. 1. A ballistic chaotic left-right symmetric cavity connected to two leads supporting one channel each.

II. CHAOTIC SCATTERING BY REFLECTION SYMMETRIC CAVITIES

The scattering problem of a ballistic cavity with left-right (LR) symmetry connected to two leads, each with N transverse propagating modes, is described by the $2N \times 2N$ scattering matrix S, which in the stationary case relates the amplitudes of the outgoing to the incoming plane waves [25]. For a system with reflection symmetry (see Fig. 1) the S matrix is block diagonal in a basis of definite parity with respect to reflections. The general structure for S is [4]

$$S = \begin{pmatrix} r & t \\ t & r \end{pmatrix}, \quad (1)$$

where r, t are the $N \times N$ reflection and transmission matrices.

The transmission coefficient, or spinless dimensionless conductance, is obtained from the S matrix,

$$T = \mathrm{tr}(tt^\dagger); \quad (2)$$

it is proportional to the conductance of the cavity,

$$G = (e^2/h)T. \quad (3)$$

Matrices with the structure (1) can be brought to the block-diagonal form,

$$S = R_0^T \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} R_0, \quad (4)$$

where R_0 is the rotation matrix,

$$R_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1_N & 1_N \\ -1_N & 1_N \end{pmatrix}, \quad (5)$$

1_n denotes the $n \times n$ unit matrix, and

$$r = (S_1 + S_2)/2, \quad (6)$$

$$t = (S_1 - S_2)/2. \quad (7)$$

Here, $S_1 = r + t$, $S_2 = r - t$ are the most general $N \times N$ scattering matrices.

In the absence of any symmetry the condition of flux conservation implies unitarity for S_j $(j = 1, 2)$: $S_j S_j^\dagger = 1_N$. In Dyson's scheme [26] this case is called "unitary" and it is designated by $\beta = 2$. In addition, in the presence of time reversal invariance S_j is symmetric: $S_j = S_j^T$. This is the "orthogonal" case, designated as $\beta = 1$. It means that two independent $N \times N$ scattering matrices S_1, S_2 unitary (and

symmetric) for $\beta = 2$ (1) generate the most general $2N \times 2N$ unitary (and symmetric) S matrix with the structure (1):

$$SS^\dagger = 1_{2N}, \quad \text{for } \beta = 2, \quad (8)$$

$$S = S^T, \quad \text{for } \beta = 1. \quad (9)$$

When the classical dynamics of the system is chaotic, most transport properties are sample specific and a statistical analysis of the quantum-mechanical problem is called for. That analysis is performed by the construction of ensembles of physical systems, described mathematically by ensembles of S matrices distributed according to a probability law. The starting point is a uniform distribution where the scattering matrix is a member of one of the *circular ensembles*, defined by the *invariant measure* which is the precise formulation of the intuitive notion of *equal a priori probabilities* in the space of scattering matrices. In our case S_1 and S_2 are independent of each other, statistically uncorrelated, and distributed according to the invariant measure defined by

$$d\mu^{(\beta)}(S_j) = d\mu^{(\beta)}(U_0 S_j V_0), \quad (j = 1, 2). \quad (10)$$

Here, U_0, V_0 are arbitrary but fixed unitary matrices for $\beta = 2$, while $V_0 = U_0^T$ for $\beta = 1$. Equation (10) defines the circular unitary (orthogonal) ensemble, CUE (COE), for $\beta = 2$ ($\beta = 1$) [27].

Then, the invariant measure for S with the structure (1) can be written as [3,4]

$$d\hat{\mu}^{(\beta)}(S) = d\mu^{(\beta)}(S_1) d\mu^{(\beta)}(S_2). \quad (11)$$

S matrices of the form given by Eq. (1), which satisfy Eq. (8) are appropriate for systems with reflection symmetry in the absence of TRI. With the additional condition (9) it is appropriate for LR systems in the presence of TRI. However, when TRI is broken by a uniform magnetic field, the problem of LR-symmetric cavities is mapped to one of asymmetric cavities [4] with $\beta = 1$ with t replaced by r. We are interested in the case when TRI is not broken by a magnetic field [24]. Other spatial symmetries [4] will not be considered here.

The parametric derivative of S_j $(j = 1, 2)$ with respect to the energy of incidence can be defined in terms of a symmetrized form of the Wigner-Smith time delay matrix, whose eigenvalues are also the proper delay times. The derivative of S with respect to another external parameter X is defined in a similar way. We have [28]

$$\frac{\partial S_j}{\partial E} = i\, S_j^{1/2}\, Q_{jE}\, S_j^{1/2}, \quad \frac{\partial S_j}{\partial X} = i\, S_j^{1/2}\, Q_{jX}\, S_j^{1/2}. \quad (12)$$

Q_{jE}, Q_{jX} are $N \times N$ Hermitian matrices for $\beta = 2$, real symmetric for $\beta = 1$. The eigenvalues of Q_{jE} are $\hbar^{-1}$ times the proper delay times.

For classically chaotic cavities the joint distribution of S_j, Q_{jE}, Q_{jX} is [28]

$$P_j^{(\beta)}(S_j, Q_{jE}, Q_{jX})$$

$$\propto (\det Q_{jE})^{-2\beta N - 3(1-\beta/2)} \exp\left\{ -\beta \operatorname{tr}\left[\frac{\pi}{\Delta} Q_{jE}^{-1} \right.\right.$$

$$\left.\left. + \left(\frac{\pi X_\beta}{2\Delta} Q_{jE}^{-1} Q_{jX} \right)^2 \right] \right\}, \tag{13}$$

where Δ is the mean level spacing, and X_β is a typical scale for X which is not a universal quantity. S_j is independent of Q_{jE} and Q_{jX} and it is uniformly distributed in the space of scattering matrices, i.e., according to the invariant measure $d\mu^{(\beta)}(S_j)$. Q_{jX} is distributed as a Gaussian with a width set by Q_{jE}. The rates $\{x_{j_n} = 1/\tau_{j_n}\}$ $(n=1,\dots,N)$, where the dimensionless time delay τ_{j_n} is $\Delta/2\pi$ times the nth eigenvalue of Q_{jE}, are distributed according to the Laguerre ensemble, namely [28]

$$P_\beta(\{x_{j_n}\}) \propto \prod_{a<b} |x_{j_a} - x_{j_b}|^\beta \prod_c x_{j_c}^{\beta N/2} e^{-\beta x_{j_c}/2}. \tag{14}$$

For $N=1$ it reproduces the result of Refs. [16,18,29].

Let denote by q any of the two parameters E or X. A convenient parametrization for S_j and $\partial S_j / \partial q$ $(q=E, X)$ is

$$S_j = U_j V_j, \quad \frac{\partial S_j}{\partial q} = i\, U_j Q_{j_q} V_j, \tag{15}$$

where U_j, V_j are the most general $N \times N$ unitary matrices for $\beta=2$, while $V_j = U_j^T$ for $\beta=1$.

We parametrize Q_{jX} and Q_{jE} as [28]

$$Q_{jX} = \frac{1}{X_\beta} \Psi_j^{-1\dagger} K_j \Psi_j^{-1}, \tag{16}$$

$$Q_{jE} = \frac{2\pi}{\Delta} \Psi_j^{-1\dagger} \Psi_j^{-1}, \tag{17}$$

where K_j is a $N \times N$ Hermitian matrix for $\beta=2$, real symmetric for $\beta=1$, that have a Gaussian distribution with zero mean and variance,

$$\langle (K_j)_{ab} (K_j)_{cd} \rangle = \begin{cases} 4\, \delta_{ad}\delta_{bc} & \beta=2 \\ 4(\delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd}) & \beta=1 \end{cases}, \tag{18}$$

as can be seen by substitution of Eqs. (16) and (17) into Eq. (13). Ψ is a $N \times N$ matrix, complex in the unitary case, real in the orthogonal one. We define $Q_j = \Psi_j^{-1\dagger} \Psi_j^{-1}$ and diagonalize it,

$$Q_j = W_j \hat{\tau}_j W_j^\dagger. \tag{19}$$

The reciprocals of the eigenvalues $\{\tau_{jn}\}$ of $\hat{\tau}_j$ are distributed according to Eq. (14). The matrix of eigenvectors, W, is uniformly distributed in the unitary (orthogonal) group for $\beta=2$ $(\beta=1)$.

III. PARAMETRIC CONDUCTANCE VELOCITY DISTRIBUTIONS

A. $N=1$ Case

In this section we calculate the distributions of the partial derivatives of the dimensionless conductance, namely $\partial T / \partial E$

and $\partial T / \partial X$, when the leads support one propagating mode.

For $N=1$ the S matrix is 2×2 and $S_j = e^{i\theta_j}$ $(j=1,2)$ is uniformly distributed in the unit circle, i.e., $d\mu^{(\beta)}(S_j) = d\theta_j / 2\pi$; S is distributed as Eq. (11),

$$d\hat{\mu}^{(\beta)}(S) = \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi}. \tag{20}$$

The transmission coefficient is

$$T = \frac{1}{2}[1 - \cos(\theta_1 - \theta_2)], \tag{21}$$

whose probability distribution is given by [3]

$$w_\beta(T) = \frac{1}{\pi\sqrt{T(1-T)}}, \tag{22}$$

for any β.

The partial derivative with respect to the external parameter q $(q=E, X)$ is

$$\frac{\partial T}{\partial q} = \frac{1}{2}\sin(\theta_1 - \theta_2)\left(\frac{\partial \theta_1}{\partial q} - \frac{\partial \theta_2}{\partial q} \right); \tag{23}$$

from Eq. (21) we rewrite the last equation as

$$\frac{1}{\sigma\sqrt{T(1-T)}} \frac{\partial T}{\partial q} = \left(\frac{\partial \theta_1}{\partial q} - \frac{\partial \theta_2}{\partial q} \right), \tag{24}$$

where $\sigma = \pm 1$ depending on the phases of the 1×1 S_1, S_2 matrices, θ_1, θ_2. A similar expression to Eq. (24) holds for asymmetric cavities [19] where $\partial T / \partial E$ depends on T but the ratio $(\partial T / \partial E)/\sqrt{T(1-T)}$ is independent of T. However, T is linearly uncorrelated with $\partial T / \partial q$ $(q=E,X)$ (see Sec. III B).

We write Eq. (23) as

$$\frac{\partial T}{\partial q} = \frac{z_q}{2}\sin(\theta_1 - \theta_2), \tag{25}$$

where we have defined

$$z_q = \frac{\partial \theta_1}{\partial q} - \frac{\partial \theta_2}{\partial q}. \tag{26}$$

From Eq. (12), $\partial \theta_j / \partial E = \tau_j$. In a similar way we denote by $\rho_j = \partial \theta_j / \partial X$. Then,

$$z_E = \tau_1 - \tau_2, \quad z_X = \rho_1 - \rho_2. \tag{27}$$

The sets $(\theta_1, \tau_1, \rho_1)$, $(\theta_2, \tau_2, \rho_2)$ are statistically independent, while the joint distribution of θ_j, τ_j, ρ_j $(j=1,2)$ is [Eq. (13)]

$$P_j^{(\beta)}(\theta_j, \tau_j, \rho_j) = C_\beta \tau_j^{-3-\beta/2} \exp\left(-\frac{\alpha_\beta}{\tau_j} + \frac{\alpha_\beta^2 X_\beta^2 \rho_j^2}{4\beta} \frac{1}{\tau_j^2} \right), \tag{28}$$

where $\alpha_\beta = \beta\pi/\Delta$, $C_\beta = \alpha_\beta^{(2+\beta/2)} X_\beta/2\sqrt{\beta\pi}\,\Gamma(1+\beta/2)$, and $\Gamma(x)$ is the gamma function.

The distribution of $\partial T / \partial q$ $(q=E,X)$ is

$$P_\beta(\partial T/\partial q) = \int_0^{2\pi} \frac{d\theta_1}{2\pi} \int_0^{2\pi} \frac{d\theta_2}{2\pi} \int_{-\infty}^{\infty} dz_q P'_\beta(z_q)$$

$$\times \delta\left[\frac{\partial T}{\partial q} - \frac{z_q}{2}\sin(\theta_1 - \theta_2)\right], \quad (29)$$

which can be written as

$$P_\beta(\partial T/\partial q) = \frac{2}{\pi}\int_0^\pi P'_\beta\left(\frac{2}{\sin\theta}\frac{\partial T}{\partial q}\right)\frac{d\theta}{|\sin\theta|}, \quad (30)$$

or, equivalently,

$$P_\beta(\partial T/\partial q) = \frac{8}{\pi}\int_{2|\partial T/\partial q|}^{\infty} \frac{P'_\beta(z_q)}{\sqrt{z_q^2 - 4(\partial T/\partial q)^2}} dz_q, \quad (31)$$

where

$$P'_\beta(z_q) = \int_0^{2\pi} \frac{d\theta_1}{2\pi}\int_0^\infty d\tau_1 \int_{-\infty}^\infty d\rho_1 P_1^{(\beta)}(\theta_1, \tau_1, \rho_1)$$

$$\times \int_0^{2\pi} \frac{d\theta_2}{2\pi}\int_0^\infty d\tau_2 \int_{-\infty}^\infty d\rho_2 P_2^{(\beta)}(\theta_2, \tau_2, \rho_2)$$

$$\times \delta[z_q - (\xi_1 - \xi_2)], \quad (32)$$

with $\xi = \tau$ or ρ. Using Eq. (28) we see that the integrations with respect to θ_1, θ_2 are equal to 1. For the particular case of $P_\beta(\partial T/\partial E)$ the ρ variables are easily integrated, as well as one of the τ variables, say τ_2, due to the delta function factor, thus only remaining to be done the integral with respect to τ_1. After this is done and using Eq. (30) we get that

$$P_\beta(\partial T/\partial E)$$

$$= A_\beta\left|\frac{\partial T}{\partial E}\right|^{-(2+\beta)}\int_0^\pi d\theta\,(\sin\theta)^{(1+\beta/2)}$$

$$\times \int_0^\infty dv \frac{e^{-v}\left[\left|\frac{\partial T}{\partial E}\right|\frac{v}{\alpha_\beta} + f_\beta\left(v, \theta, \frac{\partial T}{\partial E}\right) - \sin\theta\right]^{2+\beta}}{f_\beta\left(v, \theta, \frac{\partial T}{\partial E}\right)\left[\left|\frac{\partial T}{\partial E}\right|\frac{v}{\alpha_\beta} + f_\beta\left(v, \theta, \frac{\partial T}{\partial E}\right)\right]^{1+\beta/2}}, \quad (33)$$

where $f_\beta(v, \theta, \partial T/\partial E) = \{[(\partial T/\partial E)v/\alpha_\beta]^2 + \sin^2\theta\}^{1/2}$ and $A_\beta = \alpha_\beta^{(1+\beta)}/8\beta\pi[\Gamma(1+\beta/2)]^2$.

For $P(\partial T/\partial X)$ one of the ρ's, say ρ_2, is easily integrated due to the delta function factor, giving a Gaussian integral for ρ_1 which is also easily obtained. The result is substituted in Eq. (31), the order of integration is interchanged to first perform the z_X integration, using an integral representation of the zero order modified Bessel function $K_0(x)$. The remaining integrals, with respect to τ_1, τ_2, are transformed to new variables to obtain that
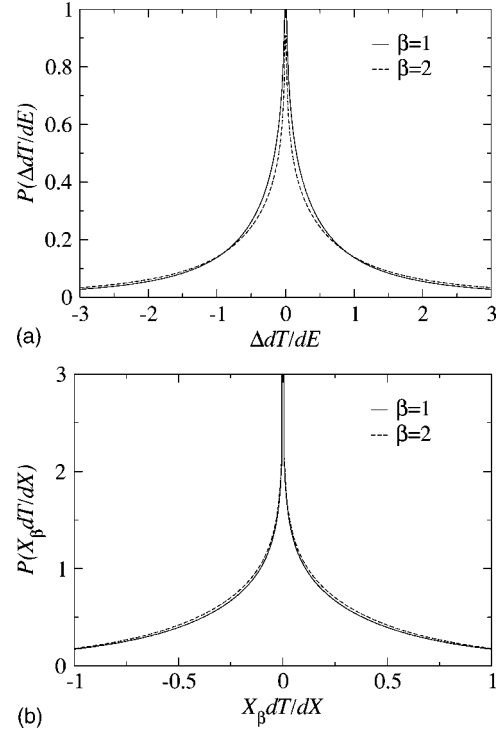

FIG. 2. Distribution of (a) $\Delta \partial T/\partial E$ and (b) $X_\beta \partial T/\partial X$ for the symmetries $\beta = 1, 2$, obtained by numerical integrations of Eqs. (33) and (34).

$$p_\beta(\partial T/\partial X)$$

$$= B_\beta \int_0^1 dx \frac{(1-x)^{(1+\beta/2)}}{x^{(2+\beta/2)}} e^{-\alpha_\beta(1-x)/x}$$

$$\times \int_0^1 dy \frac{(1-y)^{(1+\beta/2)}}{y^{(2+\beta/2)}\sqrt{(1-x)^2 y^2 + x^2(1-y)^2}} e^{-\alpha_\beta(1-y)/y}$$

$$\times \exp\left[-g_\beta(x,y)\left(\frac{\partial T}{\partial X}\right)^2\right] K_0\left[g_\beta(x,y)\left(\frac{\partial T}{\partial X}\right)^2\right], \quad (34)$$

where $B_\beta = 2X_\beta \alpha_\beta^{(3+\beta)}/\pi\sqrt{\beta\pi}[\Gamma(1+\beta/2)]^2$ and

$$g_\beta(x,y) = \frac{\alpha_\beta^2 X_\beta^2}{2\beta}\frac{(1-x)^2(1-y)^2}{(1-x)^2 y^2 + x^2(1-y)^2}. \quad (35)$$

In Fig. 2 we show the result of the numerical integration of Eqs. (33) and (34). We observe a slight dependence on β, while the T distribution is independent of it [see Eq. (22)]. In our case, a symmetric one, the dependence on β is not as clearly marked as that of the asymmetric case. Also, in contrast to the asymmetric case [19], $P_\beta(\partial T/\partial X)$ diverges logarithmically at zero transmission velocity not only for $\beta = 1$ but also for $\beta = 2$.

For large values of $|\partial T/\partial q|$ the tails are algebraic, namely

$$P_\beta(\partial T/\partial q) \propto |\partial T/\partial q|^{-2-\beta/2} \quad (36)$$

which again is in contrast with the asymmetric case where the exponent of the algrabraic tails [19] is $-2-\beta$.

B. Arbitrary N

For arbitrary N we are not able to calculate analytically the whole distribution of $\partial T/\partial q (q=E,X)$. Instead, we calculate the first two moments of the distribution analytically for any N and complement our results with a numerical simulation of $P_\beta(\partial T/\partial q)$ for $N=5, 10$. Also, we calculate the covariance between the energy and parametric derivatives, as well as the correlation of each one with the transmission coefficient. The explicit analytical calculations are presented in the Appendix.

Due to Eq. (4) the transmission coefficient depends on the interference term $S_1 S_2^\dagger$ [see Eq. (A1)], i.e., linearly on S_1 and S_2, that are independent from each other, but are equally distributed random matrices. As a result we have (see the Appendix) $\langle T(\partial T/\partial q)\rangle=0$ and $\langle(\partial T/\partial E)(\partial T/\partial X)\rangle=0$ for all N. This means that T, $\partial T/\partial E$, $\partial T/\partial X$ are linearly uncorrelated variables. However, in general they are correlated (see the Appendix) as happens in the asymmetric case [19].

We find (Appendix) that $P_\beta(\partial T/\partial q)$ has a zero mean, i.e., $\langle \partial T/\partial q\rangle=0$, as expected because $P_\beta(\partial T/\partial q)$ is an even function of $\partial T/\partial q$ (see Fig. 2). For $N=1$, which is the case of Sec. III A, $\langle(\partial T/\partial q)^2\rangle$ diverges in both cases ($\beta=1,2$). For $N>1$ the variance of $\partial T/\partial X$ is given by

$$\langle(\partial T/\partial X)^2\rangle = \frac{1}{X_1^2}\frac{2(N^2+N+2)}{(N-2)(N+1)^2} \tag{37}$$

for $\beta=1$, and

$$\langle(\partial T/\partial X)^2\rangle = \frac{1}{X_2^2}\frac{N^2+1}{N(N^2-1)} \tag{38}$$

for $\beta=2$; for both cases

$$\langle(\partial T/\partial E)^2\rangle = \frac{\pi^2 X_\beta^2}{\Delta^2}\frac{1}{N}\langle(\partial T/\partial X)^2\rangle. \tag{39}$$

We see that $\langle(\partial T/\partial q)^2\rangle$ diverges only for $\beta=1$ when $N=2$.

For large N cases we resort to numerical simulations. The results for $N=5, 10$ are presented in Fig. 3 for $\partial T/\partial X$, and in Fig. 4 for $\partial T/\partial E$ where we also include the Gaussian distributions with variances obtained from our results given by Eqs. (37) and (38) for $P_\beta(\partial T/\partial X)$, and the corresponding ones for $P_\beta(\partial T/\partial E)$. We observe that the distribution tends towards a Gaussian as N is increased as expected (see Ref. [19] and references therein).

As mentioned in a previous publication [31], for chaotic cavities we implement numerical simulations by generating 10^4 scattering matrices for each S_j ($j=1,2$) given by the Heidelberg approach [30] with $M=100$ poles. Details can be found in Ref. [31].

IV. CONCLUSIONS

We have studied the effect of spatial reflection symmetry on the distribution of the parametric derivative, $\partial T/\partial X$, of the dimensionless conductance T of a chaotic cavity, with respect to an external parameter X. In particular we considered a left-right ballistic chaotic that has two leads symmetrically placed, each one supporting N propagating modes, in the presence (absence) $\beta=1$ ($\beta=2$) of time-reversal invariance
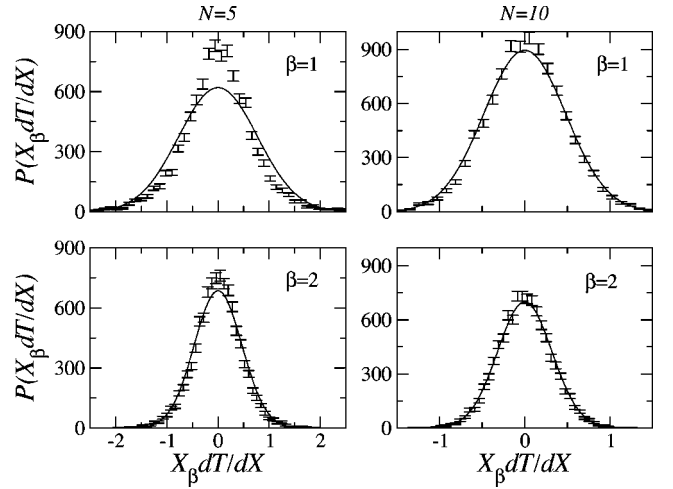

FIG. 3. Distribution of $\partial T/\partial X$ for $N=5, 10$. When N increases the distribution tends to a Gaussian distribution with variance given by Eq. (37) for $\beta=1$ and Eq. (38) for $\beta=2$.

(TRI). The external parameter could be, for instance, the shape of the cavity in the microwave technique, where our results can be tested [22].

For the particular case $N=1$, we calculated the distribution $P_\beta(\partial T/\partial X)$ by reducing the problem to quadratures and performing a numerical integration. While the distribution of T does not depend on β, the distribution of $\partial T/\partial X$ shows a slight dependence on β that is not as clearly marked as in the asymmetric case. Also, in our case, $P_\beta(\partial T/\partial X)$ diverges logarithmically at zero transmission velocity for any β. This is in contrast with the asymmetric case where for $\beta=1$ there is a logarithmic divergence and a cusp $\beta=2$. We found that $P_\beta(\partial T/\partial X)$ has algebraic tails with an exponent $-2-\beta/2$ different from that in the asymmetric case.

We found that $\partial T/\partial q$ ($q=E,X$) depends on T but the ratio $(\partial T/\partial q)/\sqrt{T(1-T)}$ is T independent as happens in the asymmetric case. We showed that T and $\partial T/\partial q$ are linearly uncorrelated for any N, although in general they are correlated.

We performed numerical simulations for $N=5, 10$ to obtain $P_\beta(\partial T/\partial X)$ and compare with a Gaussian distributions
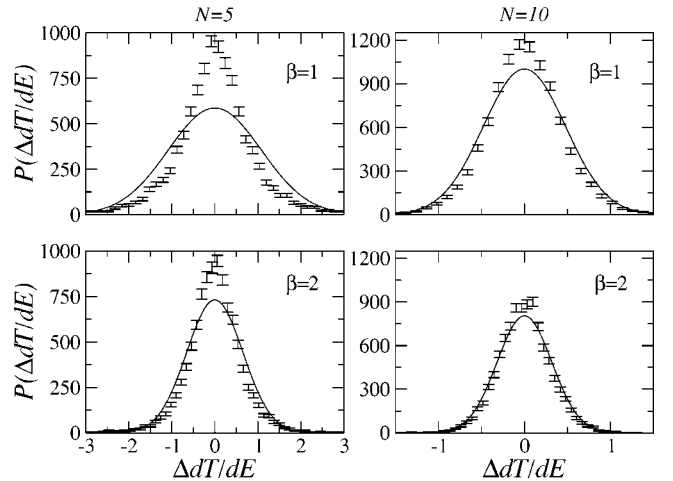

FIG. 4. As in Fig. 3 the distribution of $\partial T/\partial E$ tends to a Gaussian distribution with variance given by Eq. (39).

with variances calculated with our analytical result; we see that the distribution of the parametric derivative of T tends towards a Gaussian distribution when N becomes large.

Finally, we emphasize that our work is concerned with fully chaotic ballistic cavities. Our results are consistent with both experimental findings and simulations [22]. It would be also very interesting to single out the contribution of regular orbits (see [32–36] and references therein) something that is beyond the scope of the present work, since it would require to modify the method here employed.

APPENDIX: MEAN AND VARIANCE OF $\partial T/\partial q$ ($q=E$, x)

From Eqs. (2) and (7) we see that

$$T = \frac{N}{2} - \frac{1}{2}\sum_{a,b=1}^{N} \text{Re}\,(S_1)_{ab}(S_2^*)_{ab}, \qquad (A1)$$

where we have used the unitarity condition for S_1 and S_2. Because S_1 and S_2 are independent but equally distributed random matrices with random phases, the average of T is $\langle T\rangle = N/2$ for both symmetries ($\beta=1$ and 2), reflecting the spatial symmetry of the cavity.

Let q be one of the two parameter E or X in dimensionless units. For the same reason as in the last paragraph, the average of the q derivative of T,

$$\frac{\partial T}{\partial q} = -\frac{1}{2}\sum_{a,b=1}^{N} \text{Re}\left[(S_1)_{ab}\frac{\partial(S_2^*)_{ab}}{\partial q} + \frac{\partial(S_1)_{ab}}{\partial q}(S_2^*)_{ab}\right] \quad (A2)$$

is $\langle\partial T/\partial q\rangle = 0$.

In a similar way, after averaging the product of Eqs. (A1) and (A2) we get

$$\left\langle T\frac{\partial T}{\partial q}\right\rangle = \frac{1}{4}\sum_{a,b,a',b'=1}^{N} \text{Re}\langle(S_1)_{ab}(S_1^*)_{a'b'}\rangle\left\langle(S_2^*)_{ab}\frac{\partial(S_2)_{a'b'}}{\partial q}\right\rangle. \qquad (A3)$$

The first term on the right hand side is real positive. For $q=E$, the second term of Eq. (A3) is pure imaginary as can be seen by substitution of Eqs. (15), (17), and (19) and averaging over the unitary matrices U_2, V_2, and W_2 for $\beta=2$, or over U_2 and W_2 for $\beta=1$. For $q=X$ the parametrization (15) and Eq. (16) give a result linear on the matrix K_2 which has zero mean. Therefore $\langle T(\partial T/\partial q)\rangle = 0$. Following the same procedure we expect that $\langle T^m(\partial T/\partial q)\rangle = 0$ with m an integer (we have verified also the case $m=2$). I.e., T^m is linearly uncorrelated with $\partial T/\partial E$ and $\partial T/\partial X$. However, in general they are correlated; for instance we have checked that $\langle T(\partial T/\partial q)^2\rangle \neq 0$.

Following the same arguments we get

$$\left\langle\frac{\partial T}{\partial q}\frac{\partial T}{\partial q'}\right\rangle = \text{Re}\,\frac{1}{4}\sum_{a,b,a',b'=1}^{N}\left[\langle(S_1)_{ab}(S_1^*)_{a'b'}\rangle\right.$$

$$\times\left\langle\frac{\partial(S_2^*)_{ab}}{\partial q}\frac{\partial(S_2)_{a'b'}}{\partial q'}\right\rangle + \left\langle(S_1)_{ab}\frac{\partial(S_2^*)_{a'b'}}{\partial q'}\right\rangle$$

$$\times\left.\left\langle\frac{\partial(S_2^*)_{ab}}{\partial q}(S_2)_{a'b'}\right\rangle\right], \qquad (A4)$$

where q and q' denote E or X independently. Also we have used that S_1 and S_2 are unitary $\beta=2$ and symmetric $\beta=1$ independent and equally distributed random matrices. When $q=E$ and $q'=X$ the result is $\langle(\partial T/\partial E)(\partial T/\partial X)\rangle = 0$ because the partial derivative with respect to X is linear in the matrix K_j which has zero mean. Then, $\partial T/\partial E$ and $\partial T/\partial X$ are linearly uncorrelated. Again, in general they are correlated. For instance, $\langle(\partial T/\partial E)^2(\partial T/\partial X)^2\rangle \neq 0$.

Let us consider $q=q'=X$. Using the parametrization (15) and averaging over U_j and V_j for $\beta=2$, or over U_j for $\beta=1$, using the results of Ref. [37] we arrive to

$$\langle(S_1)_{ab}(S_1^*)_{a'b'}\rangle = \begin{cases} (\delta_a^{a'}\delta_b^{b'} + \delta_a^{b'}\delta_b^{a'})/(N+1) & \beta=1 \\ \delta_a^{a'}\delta_b^{b'}/N & \beta=2 \end{cases}$$

$$(A5)$$

and

$$\left\langle(S_j)_{ab}\frac{\partial(S_j^*)_{a'b'}}{\partial X}\right\rangle = 0 \quad (j=1,\,2) \qquad (A6)$$

for both $\beta=1$, and 2, where we used $\langle Q_{X\alpha\alpha}\rangle = 0$ because K_j in the parametrization (16) has zero mean; while

$$\left\langle\frac{\partial(S_2^*)_{ab}}{\partial X}\frac{\partial(S_2)_{a'b'}}{\partial X}\right\rangle = \frac{(\delta_a^{a'}\delta_b^{b'} + \delta_a^{b'}\delta_b^{a'})}{N(N+1)}\sum_{\alpha=1}^{N}\langle(Q_X^2)_{\alpha\alpha}\rangle$$

$$(A7)$$

for $\beta=1$,

$$\left\langle\frac{\partial(S_2^*)_{ab}}{\partial X}\frac{\partial(S_2)_{a'b'}}{\partial X}\right\rangle = \frac{\delta_a^{a'}\delta_b^{b'}}{N^2}\sum_{\alpha=1}^{N}\langle(Q_X^2)_{\alpha\alpha}\rangle \qquad (A8)$$

for $\beta=2$. Using Eqs. (16)–(18), Eqs. (A7) and (A8) can be written as

$$\left\langle\frac{\partial(S_2^*)_{ab}}{\partial X}\frac{\partial(S_2)_{a'b'}}{\partial X}\right\rangle = 4\frac{(\delta_a^{a'}\delta_b^{b'} + \delta_a^{b'}\delta_b^{a'})}{N(N+1)}\left[\sum_{\alpha=1}^{N}\langle(Q^2)_{\alpha\alpha}\rangle\right.$$

$$\left. + \sum_{\alpha,\beta=1}^{N}\langle Q_{\alpha\alpha}Q_{\beta\beta}\rangle\right] \qquad (A9)$$

for $\beta=1$ and

$$\left\langle\frac{\partial(S_2^*)_{ab}}{\partial X}\frac{\partial(S_2)_{a'b'}}{\partial X}\right\rangle = 4\frac{\delta_a^{a'}\delta_b^{b'}}{N^2}\sum_{\alpha,\beta=1}^{N}\langle Q_{\alpha\alpha}Q_{\beta\beta}\rangle \quad (A10)$$

for $\beta=2$.

We substitute Eqs. (A5), (A6), (A9), and (10) into Eq. (A4) for $q=q'=X$. After we sum over a, a', b, b' the result is

$$\langle (\partial T/\partial X)^2 \rangle = \frac{2}{N+1} \left[\sum_{\alpha=1}^{N} \langle (Q^2)_{\alpha\alpha} \rangle + \sum_{\alpha,\beta}^{N} \langle Q_{\alpha\alpha} Q_{\beta\beta} \rangle \right]$$

(A11)

for $\beta=1$,

$$\langle (\partial T/\partial X)^2 \rangle = \frac{1}{N} \sum_{\alpha,\beta}^{N} \langle Q_{\alpha\alpha} Q_{\beta\beta} \rangle$$

(A12)

for $\beta=2$.

Now, the dialgonalization of Q, Eq. (19), and the results of Ref. [37] lead us to

$$\langle (\partial T/\partial X)^2 \rangle = \frac{2N}{N+1} [2\langle \tau_1^2 \rangle + (N-1)\langle \tau_1 \tau_2 \rangle]$$ (A13)

for $\beta=1$ and

$$\langle (\partial T/\partial X)^2 \rangle = \langle \tau_1^2 \rangle + (N-1)\langle \tau_1 \tau_2 \rangle$$ (A14)

for $\beta=2$.

Finally, the average with respect to the τ variables are done using Eq. (14). When $N=1$ only the first term of Eqs. (A13) and (A14) contribute. But, $\langle \tau_1^2 \rangle$ diverges for $\beta=1$ and $\beta=2$ and then $\langle (\partial T/\partial X)^2 \rangle$. For $N>1$ we have

$$\langle \tau_1^2 \rangle = \begin{cases} 2N!/(N-2)(N+1)! & \beta=1 \\ 2N(N-2)!/(N+1)! & \beta=2 \end{cases}$$ (A15)

and

$$\langle \tau_1 \tau_2 \rangle = (N-1)!/(N+1)! \quad \beta=1,2.$$ (A16)

The final result for $\langle (\partial T/\partial X)^2 \rangle$ is given by Eqs. (37) and (38) for $\beta=1$ and $\beta=2$, respectively. In a similar way we arrive at the result given by Eq. (39) for the variance of $\partial T/\partial E$.

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